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Asymptotic behaviour of the magnetic field in a perfectly conducting chaotic flow

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Abstract. Magnetic fields in perfectly conducting chaotic flows are known to exhibit an arbitrarily fine-scaled alternation in the orientation of these fields. As a result, there exists a degree of cancellation when integrating the field in any open set. We show that when this cancellation is enough to keep bounded some generalized variation of the field, it behaves rather regularly asymptotically in time, despite the fact that it tends to concentrate in a fractal set. The magnetic field, divided by the total magnetic moment, converges in a weak sense, and several means of its vector potential converge pointwise.

1. Introduction

Kinematic dynamo theory addresses the problem of the evolution of the magnetic field generated by the motions of a conducting flow without taking into account the effect of this field upon the fluid velocity through the Lorentz force. As such it is adequate for the study of small fields, or the initial stages of a rapidly evolving seed magnetic field, such as those occurring in the so-called fast dynamos, which have been extensively studied [1]. It is known that fast dynamos are only possible in chaotic flows [2], which makes doubly relevant the work of Ott and co-workers [3–5] for the comprehension of the magnetohydrodynamics of these flows. These authors, as well as many others, use maps to visualize the evolution of the magnetic field under the induction equation

$$\frac{\partial \boldsymbol{B}}{\partial t} = \operatorname{curl}(\boldsymbol{u} \times \boldsymbol{B} - \eta \operatorname{curl} \boldsymbol{B}). \tag{1}$$

It is known that when the magnetic diffusivity η is taken as zero, the field is frozen in the fluid, i.e. it is transported by it as material points. In this case the map consists simply of transport by the fluid flow. When diffusivity exists, a two-step map is used: first translation by the flow, and then static diffusion to account for the effects of resistivity. That this procedure adequately mimics the real kinematic evolution of the field is proved, for instance, in [2]. Hence, this map provides an approximate solution of the induction equation,which is Ott's procedure: field evolution is a consequence of the geometry of the fluid motion. In the case of a perfectly conducting fluid, the magnetic field tends to be concentrated in a zero-volume fractal set [3, 4]: Thus, $\boldsymbol{B}(t)/\|\boldsymbol{B}(t)\|_\infty$ tends to zero when $t \to \infty$ in any integral norm, where $\|\boldsymbol{B}(t)\|_\infty$ stands for the supremum norm. What remains to be seen is if $\boldsymbol{B}$ itself has a limit in some sense when $t \to \infty$ or its behaviour in time is too irregular to prescribe any stability. We intend showing that for any sequence $t_n \to \infty$, there exists a subsequence (t_{j_n}) such that $\boldsymbol{B}(t_{j_n})$, divided by the total magnetic moment $|\iint_V \boldsymbol{B}(t_{j_n})\,\mathrm{d}V|$, has a limit in a certain weak sense. Moreover, this limit is rather regular as a space function. Hence, it may occur that the magnetic moment tends to infinity (as it happens in fast dynamos), to tend to zero or not to have a limit, but the mean $\boldsymbol{B}/|\iint_V \boldsymbol{B}\,\mathrm{d}V|$ varies within a set of limit states which are not too irregular.

It is known [4, 5] that the field tends to point in opposite directions in very close parts of the domain V. To evaluate the oscillation of the field, Du and Ott [4, 5] introduced the concept of cancellation exponent, which is the exponent κ such that when dividing V into boxes $V_1, \ldots, V_N$ of edge size ε, we have

$$\sum_{i=1}^{N}\left|\iint_{V_i}\boldsymbol{B}\,\mathrm{d}V\right| = \varepsilon^{-3\kappa} f(\varepsilon)\left|\iint_{V}\boldsymbol{B}\,\mathrm{d}V\right| \tag{2}$$

with $f(\varepsilon) \to 1$ as $\varepsilon \to 0$. (The cancellation exponent is 3κ in Du and Ott's notation.) They presented a formula for the fast dynamo growth [5] as a function of the cancellation and the Lyapunov exponents of the flow and considered the case with positive magnetic diffusion as a smoothing of the ideal field in the scale of the square root of the diffusion, so that the previous results remain valid for adequate intervals of time and lengthscale.

The dynamo growth formula may need some refinements [1]; for certain flows it does not work [6]. Also, the diffusive field is not a simple smoothing of the ideal one [7]. Even the existence of the cancellation exponent is difficult to prove. The numerical evidence for some classical models such as the baker's map and the stretch–fold–shear dynamo points to a definite value, and assuming that the exponent exists this value may be found theoretically [4], but so far there is no rigorous existence proof for any model.

Du and Ott [4] argued that the fraction of the total magnetic flux within the box V_i becomes a constant in time, to which it should be added that when one divides V into N identical parts, the sum $\sum|\iint_{V_i}\boldsymbol{B}\,\mathrm{d}V|$ must behave precisely as a power N^κ of N, i.e.

$$\sum_{i=1}^{N}\mathrm{vol}(V_i)^\kappa\left|\iint_{V_i}\boldsymbol{B}\,\mathrm{d}V\right|\bigg/\left|\iint_{V}\boldsymbol{B}\,\mathrm{d}V\right| \tag{3}$$

tends to a positive constant as $N \to \infty$. This demands a very precise and uniform behaviour of the field cancellation. We will weaken this hypothesis in the sense of admitting only that this amount must remain bounded for all time. However, if in the limit the cancellation works equally well in finer scales, the value of the integrals $|\iint_{V_i}\boldsymbol{B}\,\mathrm{d}V|$ should not differ greatly despite the size of V_i; hence their sum should be bounded by N^κ even for a non-uniform decomposition of V. We will take parallel slabs in any fixed direction given by a unit vector $\boldsymbol{w}r$. The above hypothesis may be written as follows.

Let Ω be the unit sphere in $\mathbb{R}^3$, (a, b) be an interval large enough so that in any direction $\boldsymbol{w} \in \Omega$, V is contained in $\{\boldsymbol{x} : \boldsymbol{x}\cdot\boldsymbol{w} \in (a, b)\}$. Let $a = r_0 < r_1 < \cdots < r_{N+1} = b$ be a partition of (a, b), $\boldsymbol{w}$ be a fixed unit vector, $V_i = \{\boldsymbol{x} : \boldsymbol{x}\cdot\boldsymbol{w} \in (r_i, r_{i+1}]\}$, and let $\Phi(t)$ be the total magnetic moment $|\iint_V \boldsymbol{B}(t)\,\mathrm{d}V|$.

We will say that $\boldsymbol{B}$ possesses uniform cancellation of exponent κ if for any probability measure P in (a, b),

$$\sum_{i=1}^{N} P((r_i, r_{i+1}])^\kappa\left|\iint_{V_i}\boldsymbol{B}(t)\,\mathrm{d}V\right| \leqslant \text{constant}\cdot\Phi(\text{t}) \tag{4}$$

uniformly for all $\boldsymbol{w}$, t and all partitions.

To simplify the arguments, we will assume the field $\boldsymbol{B}(0)$ at $t = 0$ to vanish at the boundary ∂V, and the flow not to surpass its boundaries; since the field lines are transported as material points (the frozen field theorem), $\boldsymbol{B}(t)$ always vanishes at ∂V. Also we take the velocity $\boldsymbol{u}$ smooth enough (which does not preclude its chaotic character). Hence, if $\boldsymbol{B}(0)$ is smooth, so is $\boldsymbol{B}(t)$ for all t, although its derivatives will tend to grow as the field becomes more and more fractal-like. We will say that the field within a chaotic flow possesses uniform cancellation of exponent κ if the above inequality holds for some constant $\kappa < 1$.

2. Uniform cancellation and generalized variation

Lemma 2.1. A field $\boldsymbol{B}$ possesses uniform cancellation of exponent κ if and only if there exists a constant K such that for every unit vector $\boldsymbol{w}$, for any partition $(r_i)_{i=1}^{N+1}$ of (a, b) and for $V_i = \{\boldsymbol{w}r \in V : r \in (r_i, r_{i+1}]\}$, we have

$$\sum_{i=1}^{N} \left| \iint_{V_i} \boldsymbol{B}(t)\,\mathrm{d}V \right|^{1/(1-\kappa)} \leqslant K\Phi(t)^{1/(1-\kappa)} \tag{5}$$

for all time.

Proof. Since we may take any probability measure P, we have that for every $(\lambda_i)_{i=1}^{N+1}$, $\sum_i \lambda_i = 1$, $\lambda_i \geqslant 0$, $\sum_i \lambda_i^{\kappa} m_i \leqslant$ constant. Now the $1/(1-\kappa)$-norm of (m_i) is the maximum of $\sum_i \mu_i m_i$, when the $1/\kappa$-norm of (μ_i) is less than 1. Since $m_i \geqslant 0$, this maximum is reached for $\mu_i \geqslant 0$. Thus, it is enough to take $\lambda_i = \mu_i^{1/\kappa}$. The reverse implication is a trivial consequence of Holder's inequality. Hereafter we will denote by p the value $1/(1-\kappa) > 1$.

Let us remember that for any solenoidal field $\boldsymbol{B}$ within the Sobolev class $H^m(\mathbb{R}^3)$ there exists a unique solenoidal field $\boldsymbol{A}$ (called a vector potential) such that curl $\boldsymbol{A} = \boldsymbol{B}$ and $\boldsymbol{A} \in H^{m+1}(\mathbb{R}^3)$ (see for example [8]). Now our magnetic field may be extended smoothly to $\mathbb{R}^3$ by $\mathbf{0}$, since it vanishes at ∂V. Thus there exists a solenoidal vector potential decaying smoothly to zero at infinite (although not necessarily confined to V). In the future we will deal exclusively with this field $\boldsymbol{A}$, and we will take for granted the bound in lemma 2.1. □

Theorem 2.2. Let $\boldsymbol{w} \in \Omega$, S_j the plane $\boldsymbol{w} \cdot \boldsymbol{x} = r_j$. Then there exists a constant M such that for every component A_l of $\boldsymbol{A}$, for every partition (r_j) of (a, b) and for all time

$$\sum_{j=0}^{N} \left| \int_{S_{j+1}} A_l(t)\,\mathrm{d}\sigma - \int_{S_j} A_l(t)\,\mathrm{d}\sigma \right|^p \leqslant M\Phi(t)^p. \tag{6}$$

Proof. Assume for simplicity that $\boldsymbol{w}$ is one of the coordinate vectors $\boldsymbol{e}_k$. Then $\mathrm{div}(\boldsymbol{A} \times \boldsymbol{e}_i) =$ curl $\boldsymbol{A} \cdot \boldsymbol{e}_i = B_i$. By the divergence theorem,

$$\int_{S_{j+1}} (\boldsymbol{A} \times \boldsymbol{e}_i) \cdot \boldsymbol{e}_k\,\mathrm{d}\sigma - \int_{S_j} (\boldsymbol{A} \times \boldsymbol{e}_i) \cdot \boldsymbol{e}_k\,\mathrm{d}\sigma = \iint_{V_j} B_i\,\mathrm{d}V. \tag{7}$$

For $i \neq k$, $\boldsymbol{e}_i \times \boldsymbol{e}_k = \pm\boldsymbol{e}_l$ (the remaining coordinate vector). Thus

$$\left| \int_{S_{j+1}} A_l\,\mathrm{d}\sigma - \int_{S_j} A_l\,\mathrm{d}\sigma \right| = \left| \iint_{V_j} B_i\,\mathrm{d}V \right|. \tag{8}$$

In this way we obtain all the components $l \neq k$. The corresponding integral for A_k is zero because div $\boldsymbol{A}$ is zero. Anyway

$$\left| \int_{S_{j+1}} \boldsymbol{A}\,\mathrm{d}\sigma - \int_{S_j} \boldsymbol{A}\,\mathrm{d}\sigma \right| \leqslant \left| \iint_{V_j} \boldsymbol{B}\,\mathrm{d}V \right|. \tag{9}$$

The consequence follows from our hypothesis that $\boldsymbol{B}$ possesses uniform cancellation. Hereafter we will denote by $\boldsymbol{\beta}$ the normalized field $\boldsymbol{B}/\Phi$, and by $\boldsymbol{\alpha}$ its vector potential $\boldsymbol{A}/\Phi$.

Recall that the integral of a function f in the plane $\boldsymbol{w}\cdot\boldsymbol{x}=r$ with the area measure $\mathrm{d}\sigma$ is called the Radon transform $Rf(\boldsymbol{w},r)$ (the notation is not standard, but more convenient for our purposes). For facts concerning the Radon transform we will follow [9]. Rf is in fact defined in the projective space $\mathbb{P}^3$; alternatively we may view functions in $\mathbb{P}^3$ as even functions in $\Omega\times\mathbb{R}$ (since the plane $\boldsymbol{w}\cdot\boldsymbol{x}=r$ is the same as the plane $-\boldsymbol{w}\cdot\boldsymbol{x}=-r$). Our previous result obviously implies the following. □

Corollary 2.3. There exists a constant M such that for any $\boldsymbol{w}\in\Omega$, for every partition (r_j) of (a,b) and for all time

$$\sum_{j=0}^{N}|R\boldsymbol{\alpha}(\boldsymbol{w},r_{j+1})-R\boldsymbol{\alpha}(\boldsymbol{w},r_j)|^p\leqslant M. \tag{10}$$

Now there is a space formed precisely by the functions satisfying this bound. Its definition goes back to Wiener [10]; they were later studied by Young [11] whose results on Stieltjes integration will be used later, and generalized by Orlicz and co-workers [12, 13]. A recent unification of many of these results may be found in [14]. Functions satisfying this bound are called functions of bounded p-variation. The supremum of $(\sum_{j=0}^{N}|f(r_{j+1})-f(r_j)|^p)^{1/p}$ when taking all possible partitions is a norm in this Banach space $\mathcal{V}^p(a,b)$. A closed subspace of this is formed by all continuous functions of bounded p-variation, $\mathcal{CV}^p(a,b)$, which includes the space of functions satisfying a Holder condition of order $1/p$, $\mathcal{C}^{1/p}(a,b)$. A deep result is that if $(1/p)+(1/q)>1$, for any $f\in\mathcal{CV}^p(a,b)$, $g\in\mathcal{V}^q(a,b)$ (or vice versa), the integral $\int_a^b f(s)\,\mathrm{d}g(s)$ exists in the sense of Riemann–Stieltjes, and

$$\left|\int_a^b f(s)\,\mathrm{d}g(s)\right|\leqslant M\|f\|_{\mathcal{V}^p}\|g\|_{\mathcal{V}^q}. \tag{11}$$

All this may be vastly generalized, but to our purposes it is enough to note that for every component α_i, $R\alpha_i\in L^\infty(\Omega,\mathcal{CV}^p(a,b))$. Hereafter we will denote by $\mathrm{d}|\boldsymbol{w}|$ the Lebesgue measure on the unit sphere Ω.

Corollary 2.4. There exists a constant M such that for any

$$F\in L^1(\Omega,\mathcal{V}^q(a,b))$$

with $(1/p)+(1/q)>1$, and for all time

$$\left|\iint_{\Omega\times\mathbb{R}}FR\boldsymbol{B}\,\mathrm{d}|\boldsymbol{w}|\,\mathrm{d}r\right|\leqslant\|F\|_{L^1(\Omega,\mathcal{V}^q(a,b))}\|R\boldsymbol{A}\|_{L^\infty(\Omega,\mathcal{CV}^p(a,b))}. \tag{12}$$

Proof. It is enough to realize that for any component B_m of $\boldsymbol{B}$ transversal to $\boldsymbol{w}$, $RB_m\,\mathrm{d}r=\pm\mathrm{d}RA_n$ for some component of $\boldsymbol{A}$. For instance, for $\boldsymbol{w}=\boldsymbol{e}_l$, $RB_i\,\mathrm{d}r=\pm\mathrm{d}RA_k$, when $\boldsymbol{e}_i\times\boldsymbol{e}_k=\pm\boldsymbol{e}_l$. The component of $\boldsymbol{B}$ in the $\boldsymbol{w}$-direction has zero Radon transform by Stokes' theorem.

This result is optimal in the sense that for $(1/p)+(1/q)\leqslant 1$, there exist functions $f\in\mathcal{CV}^p(a,b)$, $g\in\mathcal{CV}^q(a,b)$ such that $\int_a^b f(s)\,\mathrm{d}g(s)$ is divergent. These functions may even be taken within the Holder spaces $\mathcal{C}^{1/p}(a,b)$, $\mathcal{C}^{1/q}(a,b)$ [13]. We will denote by R_* the dual Radon transform (see [9]). Then we have the following. □

Corollary 2.5. When F belongs to a bounded set in $L^1(\Omega,\mathcal{V}^q(a,b))$, with $(1/p)+(1/q)>1$, the value $\iint_{\mathbb{R}^3}(R_*F)\boldsymbol{\beta}\,\mathrm{d}V$ remains bounded for all time.

Proof. This value coincides, except by a multiplicative constant, with the integral in the previous corollary by the main duality theorem of Radon transforms (see [9]).

Thus we must analyse which kind of functions are the dual Radon transforms of functions in $L^1(\Omega, \mathcal{V}^q(a, b))$. Although they do not coincide with any classical space, they contain some well known ones which will give us our theorems on the boundedness of the field. □

3. Convergence of the magnetic field

Theorem 3.1. If $\boldsymbol{B}$ possesses uniform cancellation, $\boldsymbol{\beta}$ is bounded for all time in the $H^{-2}(V)$-norm and the vector potential $\boldsymbol{\alpha}$ bounded in the $H^{-1}(V)$-norm.

Proof. We must show that $H^2(V)$ is contained in $R_* L^1(\Omega, \mathcal{V}^q(a, b))$, that is $R_*^{-1} H^2(V) \subset L^1(\Omega, \mathcal{V}^q(a, b))$. Let us see first that $RL^2(V) \subset L^2(\Omega, H^1(\mathbb{R}))$. This follows from the fact that the three-dimensional Fourier transform $\mathcal{F}_3 f(\boldsymbol{w}s)$ is the one-dimensional Fourier transform with respect to s of the Radon transform (see [9]): $\mathcal{F}_3 f(\boldsymbol{w}s) = \mathcal{F}_1 Rf(\boldsymbol{w}, s)$. Thus, by the Plancherel theorem,

$$\begin{aligned}\int_\Omega \mathrm{d}|\boldsymbol{w}| \int_{\mathbb{R}} \left|\frac{\partial Rf}{\partial r}\right|^2 \mathrm{d}r &= \int_\Omega \mathrm{d}|\boldsymbol{w}| \int_{\mathbb{R}} \left|\mathcal{F}_1 \frac{\partial Rf}{\partial r}(\boldsymbol{w}, s)\right|^2 \mathrm{d}s \\ &= \int_\Omega \mathrm{d}|\boldsymbol{w}| \int_{\mathbb{R}} |s\mathcal{F}_1 Rf(\boldsymbol{w}, s)|^2 \,\mathrm{d}s = \int_\Omega \mathrm{d}|\boldsymbol{w}| \int_{\mathbb{R}} |s\mathcal{F}_3 f(\boldsymbol{w}s)|^2 \,\mathrm{d}s \\ &= c \iint_{\mathbb{R}^3} |\mathcal{F}_3 f(\boldsymbol{x})|^2 \,\mathrm{d}\boldsymbol{x} = c \iint_{\mathbb{R}^3} |f(\boldsymbol{x})|^2 \,\mathrm{d}\boldsymbol{x}. \end{aligned} \tag{13}$$

The c accounts for the constant in passing from spherical to Cartesian coordinates. Hence the $L^2(\mathbb{R}^3)$-norm of f and the $L^2(\Omega \times \mathbb{R})$ of $(\partial Rf)/(\partial r)$ are equivalent. Since V is bounded and $Rf(\boldsymbol{w}, -)$ vanishes outside (a, b), the L^2-norm of $(\partial Rf)/(\partial r)$ is equivalent to the H^1-norm of Rf.

It is known (see again [9]) that $R_*^{-1} = \gamma R\Delta$, where Δ stands for the Laplace operator and γ is a constant. Since Δ takes $H^2(V)$ to $L^2(V)$ continuously, R_*^{-1} takes $H^2(V)$ to $L^2(\Omega, H^1(\mathbb{R}))$. In fact Rf vanishes outside (a, b). Any function within $H^1(a, b))$ has a square-integrable derivative, hence integrable and therefore has bounded variation. *A fortiori* it belongs to $\mathcal{CV}^q(a, b)$, for any $q \geqslant 1$. Since Ω has finite measure, $L^2(\Omega, \mathcal{CV}^q(a, b)) \subset L^1(\Omega, \mathcal{CV}^q(a, b))$. All the immersions are continuous. Hence

$$\left| \iint_V \boldsymbol{G} \cdot \boldsymbol{\beta} \,\mathrm{d}V \right| \leqslant M \|\boldsymbol{G}\|_{H^2(V)^3} \tag{14}$$

which means that $\boldsymbol{B}(t) : t \geqslant 0$ is bounded in $H^2(V)$. The result for $\boldsymbol{A}$ follows from the inequality

$$\|\boldsymbol{\alpha}\|_{H^{-1}(\mathbb{R}^3)^3} \leqslant \|\boldsymbol{\beta}\|_{H^{-2}(\mathbb{R}^3)^3} \tag{15}$$

which is a consequence of the fact mentioned above [8] that curl^{-1} takes continuously $H^1(\mathbb{R}^3)^3$ into $H^2(\mathbb{R}^3)^3$, by considering the adjoint operator.

This theorem is emphatically not the best possible. What we must demand for $\boldsymbol{G}$ is merely that $R_*^{-1}\boldsymbol{G} = \gamma(\partial^2/\partial r^2)R\boldsymbol{G}$ (which is another expression of the inverse) belongs to $L^1(\Omega, \mathcal{CV}^q(a, b)^3)$. Still, it will provide a partial answer to the question about the existence of a limit. □

Corollary 3.2. For every sequence $t_1, t_2, \ldots t_n \to \infty$, there exists a subsequence (t_{j_n}) such that $\boldsymbol{\beta}(t_{j_n})$ converges in the weak topology $\sigma(H^{-2}(V), H^2(V))$.

Proof. It is a simple consequence of Alaoglu's theorem on the weak compactness of the unit ball of the dual space, and the fact that since $H^2(V)$ is separable, this weak topology in the ball is metrizable.

Obviously the vector potentials $\boldsymbol{\alpha}$ converge in the $\sigma(H^{-1}(V), H^1(V))$-topology, but still this is not very intuitive; for instance, we cannot guarantee pointwise convergence. This, however, is true for the means of $\boldsymbol{\alpha}$. □

Theorem 3.3. For every Borel measure μ in Ω and every sequence $t_1, t_2, \ldots t_n \to \infty$, there exists a subsequence (t_{j_n}) such that

$$\int_\Omega R\boldsymbol{\alpha}(\boldsymbol{w}, r, t_{j_n})\,\mathrm{d}\mu(\boldsymbol{w})$$

converges at every point of (a, b).

Proof. It is a consequence of Helly's selection theorem as applied to the space $\mathcal{V}^p(a, b)$ (see [12]). The functions

$$r \to \int_\Omega R\boldsymbol{\alpha}(\boldsymbol{w}, t_n)\,\mathrm{d}\mu(\boldsymbol{w})$$

form a bounded set in $\mathcal{V}^p(a, b)$, since $\mu(\Omega) < \infty$; hence, there exists a subsequence pointwise convergent to a function of $\mathcal{V}^p(a, b)$.

If we take as μ the Dirac measure centred at a point, we would obtain pointwise convergence of $R\boldsymbol{\alpha}(\boldsymbol{w}_0, t_{j_n})$; by taking an atomic measure, we could obtain convergence of $R\boldsymbol{\alpha}$ in any countable subset of Ω. This result may be generalized as follows. Since functions in $\mathcal{V}^p(a, b)$ are bounded, the product of two functions in $\mathcal{V}^p$ lies in $\mathcal{V}^p$. Hence, we may obtain in the same fashion a convergent subsequence of $\int_\Omega \boldsymbol{G}(\boldsymbol{w}, r, t_n) R\boldsymbol{\alpha}(\boldsymbol{w}, r, t_{j_n})\,\mathrm{d}\mu(\boldsymbol{w})$ for all r, provided $\boldsymbol{G} \in L^1(\Omega, \mathcal{C}\mathcal{V}^p(a, b), \mathrm{d}\mu)^3$. Note, however, that this does not imply weak convergence in any sense of $R\boldsymbol{\alpha}(t_{j_n})$, since the subsequence depends on g and μ. Anyway the integral of $\boldsymbol{A}$ in any plane has limits when $t \to \infty$.

Finally, we will see that any of these limits is rather regular; they are the pointwise limit of their Fourier series at every point. We shall use the notion of (C, r) convergence, whose meaning may be found in [15]. □

Theorem 3.4. Let $\boldsymbol{F}(x)$ be the limit of some subsequence of

$$\int_\Omega \boldsymbol{G}(\boldsymbol{w}, r, t_n) R\boldsymbol{\alpha}(\boldsymbol{w}, r, t_{j_n})\,\mathrm{d}\mu(\boldsymbol{w}).$$

Then the Fourier series of $\boldsymbol{F}$ is (C, s) bounded for any $s \in [-1, 0)$ such that $-s < (1/p)$, and (C, r) summable to $\boldsymbol{F}(x)$ at every point for any $r > s$. In particular, the Fourier series of $\boldsymbol{F}$ converges to $\boldsymbol{F}$ at every point.

Proof. Let $((a_n, b_n))_n$ be any sequence of disjoint intervals in (a, b). Then

$$\sum_n \frac{|\boldsymbol{F}(b_n) - \boldsymbol{F}(a_n)|}{n^{s+1}} \leqslant \left(\sum_n |\boldsymbol{F}(b_n) - \boldsymbol{F}(a_n)|^p\right)^{1/p} \left(\sum_n \frac{1}{n^{p'(s+1)}}\right)^{1/p'} \tag{16}$$

where p' stands for the conjugate of p. The first factor is always bounded because $\boldsymbol{F} \in \mathcal{V}^p(a,b)^3$, and the second is bounded if $p'(s+1) > 1$, which amounts to $-s < (1/p)$. Hence $\boldsymbol{F}$ belongs to the space of the so-called functions of (n^{s+1})-bounded variation, whose Fourier series satisfy the above condition (see [16]). In particular this happens for $s = 0$ (functions of harmonic bounded variation [17]) whose Fourier series converge at every point.

As an additional result, it is also true that $\boldsymbol{F}$ may have only simple discontinuities [16]. □

4. Conclusions

We have shown that when a magnetic field evolves under the ideal induction equation for a fixed fluid velocity while keeping bounded a certain variation, the mean magnetic field $\boldsymbol{\beta}$ and its vector potential $\boldsymbol{\alpha}$ have sequential limits when $t \to \infty$ for a weak topology. This weak limit does not need to be unique, so that we cannot really speak of a limit field $\boldsymbol{\beta}(\infty)$. However, it offers some information on the asymptotic evolution of the field. Since $|\iint_V \boldsymbol{\beta}(t)\,\mathrm{d}V| = 1$, $\boldsymbol{\beta}(\infty)$ has integral 1 and cannot be 0. Hence, if the magnetic moment tends to zero, the field also vanishes in the limit: one could think that the ups and downs of the field may make for a small moment, but this does not happen unless the field itself decreases. If the magnetic moment remains bounded, the field has limits when $t \to \infty$ in the H^{-2} sense, whereas if the moment tends to infinity as it happens in the classical fast dynamo models, so does the field at every open subset of V, and at most at the total moment rate. It cannot happen that certain portions of the domain make for a large moment while others remain below the mean. Also, several spatial means of $\boldsymbol{\alpha}$ tend pointwise to their corresponding limits, and those are not too irregular. This shows how when well balanced they tend to be the ups and downs of the field.

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